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EE120
MT1

- Signal - fxn of 1+ var, usually just time
  - continuous time (CT) - $x(t)$
  - discrete time (DT) - $x[n]$
    - unit impulse - $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$
    - unit step - $u[n] = \begin{cases} 1 & n\geq 0 \\ 0 & n<0 \end{cases}$
    - $\delta[n] = u[n] - u[n-1]$
    - $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
  - transformations
    - $x(-t)$ - "flip"
    - $x(t-T)$ - "drag"
    - $x(T-t)$ - "flip then drag"
    - combine multiple to visualize
- System - input → ☐ → output
  - Properties
    - memoryless - output depends only on current input
    - causal - output depends only on current/past, no future
    - stability - BIBO; look for an input
    - linearity
      - scaling - $ax(t) \to ay(t)$
      - superposition - $x_1(t)+x_2(t) \to y_1(t)+y_2(t)$
    - time invariance - $x(t-T) \to y(t-T)$
      - test by adding a shift
  - LTI
    - Impulse Response $\delta[n] \to \boxed{LTI} \to h[n]$
      - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = (x*h)[n]$
      - convolution $\stackrel{\triangle}{=}$ $(x*h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- Convolution - visualize as flip and drag 2nd signal
  - Properties $(x*h)[n] := x[n]*h[n]$
    - $(x*\delta)[n] = x[n]$
    - $x[n]*\delta[n-N] = x[n-N]$
    - $x*h = h*x$
    - $x*(h_1+h_2) = x*h_1 + x*h_2$
    - $x*(h_1*h_2) = (x*h_1)*h_2$
  - Parallel LTI
    - $x \to \{\boxed{h_1}, \boxed{h_2}\} \to y \;::\; x \to \boxed{h_1+h_2} \to y$
  - Series LTI
    - $x \to \boxed{h_1} \to \boxed{h_2} \to y \;::\; x \to \boxed{h_1*h_2} \to y$
- LTI
  - causal iff $h[n]=0$ for $n<0$
  - stable iff $\sum_{k=-\infty}^{\infty} |h[n]| < \infty$
- CT LTI
  - Impulse $\leftarrow \delta_\Delta(t)$, $\frac{1}{\Delta}$, 0, Δ $\quad \delta(t) \stackrel{\triangle}{=} \lim_{\Delta\to 0} \delta_\Delta(t)$
    - can actually use any fxn $g(t)$ where $\int_{-\infty}^{\infty} g(t)dt = 1$; $\delta(t) = \lim_{\epsilon\to 0} g_\epsilon(t) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} g(\frac{t}{\epsilon})$
    - "distribution" T_g for x: $\langle T_g, x\rangle = \int_{-\infty}^{\infty} g(t)^* x(t) dt$
    - $f(t)\delta(t) = f(0)\delta(t)$
    - $f(t)\delta(t-T) = f(T)\delta(t-T)$
    - $\delta(at) = \frac{1}{|a|}\delta(t)$
  - Convolution Integral
    - $h(t)$ response to $\delta(t)$
    - $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
    - $(x*\delta)(t) = x(t)$
    - $x(t)*\delta(t-T) = x(t-T)$
  - causal $h(t)=0$ for $t<0$
  - stable $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$
- LTI Complex Exponential
  - $x(t) = e^{st} = e^{(a+bj)t} = \underbrace{e^{at}}_{\text{envelope}} \underbrace{e^{bjt}}_{\text{periodic}}$
  - $x[n] = z^n = (re^{j\omega})^n = r^n e^{j\omega n}$
  - $e^{st} \to \boxed{LTI} \to y(t) = H(s)e^{st}$ ⎫ "transfer functions"
  - $z^n \to \boxed{LTI} \to y[n] = H(z)z^n$ ⎭ LTI response always just scaled
  - $H(s) \stackrel{\triangle}{=} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$
  - $H(z) \stackrel{\triangle}{=} \sum_{k=-\infty}^{\infty} h[k]z^{-k}$
  - $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$
  - $H(e^{j\omega}) = \sum h[k]e^{-j\omega k}$
  - Geometric series
    $\sum_{i=0}^{n} r^n = \frac{1-r^{n+1}}{1-r}$

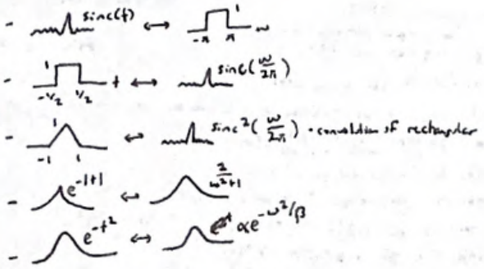
- $\cos(\theta) = \frac{1}{2}(e^{j\theta}+e^{-j\theta})$
- $\sin\theta = \frac{1}{2j}(e^{j\theta}-e^{-j\theta})$

π fastest freq in DT

- FIR (Finite impulse response)
  - $h[n]$ finite width, always stable
- IIR (Infinite impulse response)
  - $h[n]$ infinite width,
- Constant-Coefficient Linear Difference Equation
  $a_0y[n] + a_1y[n-1] + \dots a_Ny[n-N] = b_0x[n] + \dots + b_Mx[n-M]$
  - causal and LTI if $a_0\neq 0$ and $y[n]=0$ for before $x[n]$ become nonzero
  - if $a_1, \dots a_N = 0$, then FIR, IIR from feedback terms

- CT Fourier Series
  - signal periodic w/ period T if $x(t+T) = x(t)$
    - for add 2 periodic, $T = n_1T_1 = n_2T_2$; $n_1, n_2 \in \mathbb{Z}$
  - $\omega_0 = \frac{2\pi}{T}$
  - Synthesis Eqn $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
    - Conjugate Symmetry Property
      $x(t)$ has a_k, then $x^*(t)$ has $b_k = a_{-k}^*$
    - if x real, then $a_k = a_{-k}^*$
  - Analysis Eqn $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$
    - can be any period, like $-T/2$ to $T/2$
  - Dirichlet Convergence Thm - if x is piecewise continuous, piecewise continuous derivative, with fundamental period T, $\omega_0 = 2\pi/T$; $\sum_{k=-m}^{m} a_k e^{jk\omega_0 t} = x_m(t)$
    - cont. at $t=\tau \Rightarrow \lim_{m\to\infty} x_m(\tau) = x(\tau)$
    - discont. $\Rightarrow \lim_{m\to\infty} x_m(\tau) = \frac{1}{2}(x(\tau^-)+x(\tau^+))$
  - Gibbs Phenomenon - ripples near sharp changes
  - Properties $x \stackrel{FS}{\leftrightarrow} a_k$, $y \stackrel{FS}{\leftrightarrow} b_k$
    - linearity - $Ax+By \stackrel{FS}{\leftrightarrow} Aa_k+Bb_k$
    - timeshift - $x(t-t_0) \stackrel{FS}{\leftrightarrow} a_k e^{-jk\omega_0 t_0}$
    - time reversal - $x(-t) \stackrel{FS}{\leftrightarrow} a_{-k}$
    - corollary - FS of real, symmetric are real
- DT Fourier Series
  - signal periodic w/ period N if $x[n+N] = x[n]$
    - $\cos(\omega_0 n)$ periodic if $\omega_0 N = 2\pi M$
    - smallest int N, M ↗
    - use lcm for addition of signals
    - $\omega = \pi$ is highest freq
  - $\omega_0 = \frac{2\pi}{N}$
  - Synthesis Equation $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$
    - $\langle N\rangle$ any N successive integers
    - $\phi_k[n] := e^{jk\omega_0 n} \Rightarrow x[n] = \sum_{\langle N\rangle} a_k\phi_k[n]$
      - $\phi_k[n+N] = \phi_k[n]$
      - $\phi_{k+N}[n] = \phi_k[n]$
      - $\sum_{n=\langle N\rangle} \phi_k[n] = \begin{cases} N & \text{if } k=0 (\text{mod } N) \\ 0 & \text{o/w} \end{cases}$
      - $\phi_k[n]\phi_m[n] = \phi_{k+m}[n]$
  - Analysis Equation $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\omega_0 kn}$
    - $a_{N-k} = a_k^*$ if x is real
  - FS as change of basis
    - $\vec{x} = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$ $\quad \vec{\phi}_k = \begin{bmatrix} 1 \\ e^{jk\omega_0} \\ \vdots \\ e^{jk\omega_0(N-1)} \end{bmatrix}$ $k = 0, 1, \dots, N-1$
    - $\vec{x} = a_0\vec{\phi}_0 + a_1\vec{\phi}_1 + \dots + a_{N-1}\vec{\phi}_{N-1}$ ← all orthogonal
    - $a_k = \frac{1}{N}\vec{x}\cdot\vec{\phi}_k$ ⇒ equivalent to eqn
- Cool Signals
  - moving average filter (LPF) $y[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$
    - $H(e^{j\omega}) = \begin{cases} 1 & \omega=0 \\ \frac{1}{2M+1}\frac{\sin(\omega(M+1/2))}{\sin(\omega/2)} & \omega\neq 0 \end{cases}$
  - Rectangular Pulse Train $-N_1$, 0, N_1, N
    $a_k = \begin{cases} \frac{2N_1+1}{N} & k=0 \\ \frac{1}{N}\frac{\sin(k\pi(2N_1+1)/N)}{\sin(k\pi/N)} & k\neq 0 \end{cases}$
  - $x(t) = e^{-at}u(t), a>0$
    - $X(\omega) = \frac{1}{a+j\omega}$
  - ideal LPF $h(t) = \frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}t) \stackrel{FT}{\leftrightarrow}$ $-\omega_c$, ω_c, 1 $= H(j\omega)$
    - truncate and slice
  - $\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$
    - $H(j\omega) = 1/((j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n)+1)$
    - ζ - "damping ratio"
  - $x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$
    $X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$
    ← FT of periodic signal

- CT Fourier Transform (CTFT)
  - works on aperiodic signals, just make it periodic w/ period T, and $T\to\infty$, $\omega_0 = \frac{2\pi}{T}$
  - Analysis Equation $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
    - $X(\omega)|_{\omega=k\omega_0} = Ta_k$ "envelope" for a_k (make copies to make periodic)
  - Synthesis Equation $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
  - Properties $x(t) \stackrel{FT}{\leftrightarrow} X(\omega)$ and $y(t) \stackrel{FT}{\leftrightarrow} Y(\omega)$
    - linearity - $ax(t)+by(t) \stackrel{FT}{\leftrightarrow} aX(\omega)+bY(\omega)$
    - time-shift - $x(t-t_0) \stackrel{FT}{\leftrightarrow} e^{-j\omega t_0}X(\omega)$
    - conjugate symmetry - $x^*(t) \stackrel{FT}{\leftrightarrow} X^*(-\omega)$
      - if x real, $X(\omega) = X^*(-\omega) \Rightarrow$ even $|X(\omega)|$, odd $\angle X(\omega)$
    - differentiation - $\frac{dx(t)}{dt} \stackrel{FT}{\leftrightarrow} j\omega X(\omega)$
    - Time and freq scaling - $x(at) \stackrel{FT}{\leftrightarrow} \frac{1}{|a|}X(\frac{\omega}{a})$, $a\neq 0$
  - Parseval's Relation - $\int_{-\infty}^{\infty} |x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2d\omega$
  - $(x_1*x_2)(t) \stackrel{FT}{\leftrightarrow} X_1(\omega)X_2(\omega)$ - convolution property
  - partial fraction expansion
  - Derivative $-jtx(t) \stackrel{FT}{\leftrightarrow} \frac{dX(\omega)}{d\omega}$ useful for weird $X(\omega)$, reminds me derivative
    $\frac{d^kx(t)}{dt^k} \stackrel{FT}{\leftrightarrow} (j\omega)^kX(\omega)$ $\quad (-jt)^kx(t) \stackrel{FT}{\leftrightarrow} \frac{d^kX(\omega)}{d\omega^k}$
  - frequency shift $e^{j\omega_0 t}x(t) \stackrel{FT}{\leftrightarrow} X(\omega-\omega_0)$
  - multiplication property $s(t)p(t) \stackrel{FT}{\leftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\theta)P(\omega-\theta)d\theta$ ← convolution
  - $\sum_{k=0}^{N} a_k\frac{d^ky(t)}{dt^k} = \sum_{k=0}^{M} b_k\frac{d^kx(t)}{dt^k} \Rightarrow \sum_{k=0}^{N} a_k(j\omega)^kY(j\omega) = \sum_{k=0}^{M} b_k(j\omega)^kX(j\omega)$
    $\Downarrow$
    $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k}$
- Convergence of Fourier Integral (simple but conservative)
  - $\int_{-\infty}^{\infty} |x(t)|dt < \infty \Rightarrow X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ exists and is continuous
  - also $X(\omega)\to 0$ as $\omega\to\pm\infty$
  - sufficient, but not necessary
- Generalized Fourier Transform ← a.k.a. Dirac
  - $x(t) = 1 \stackrel{FT}{\leftrightarrow} X(\omega) = 2\pi\delta(\omega)$; $x(t) = \delta(t) \stackrel{FT}{\leftrightarrow} X(\omega) = 1$
  - $x(t) = e^{j\omega_0 t} \stackrel{FT}{\leftrightarrow} X(\omega) = 2\pi\delta(\omega-\omega_0)$
  - FT of periodic signals $\sum_{k=-\infty}^{\infty} a_ke^{jk\omega_0 t} \stackrel{FT}{\leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-k\omega_0)$
- DT Fourier Transform (DTFT)
  - Analysis Equation $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ ⎬ $x[n] = \sum_{k=\langle N\rangle} a_ke^{jk\frac{2\pi}{N}n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-\frac{2\pi}{N}k)$
    - $X(e^{j\omega})|_{\omega=k\cdot\frac{2\pi}{N}} = Na_k$ "envelope" for a_k (make copies to make periodic)
  - Synthesis Equation $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
    - any 2π path
    - $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$, repeats
    - converges if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$
    - $x[n] = \delta[n] \leftrightarrow X(e^{j\omega}) = 1$
    - remember π highest freq
  - Properties $\quad X(e^{j\omega})$ repeats every 2π and flip across π
    - Time Shift: $x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$
    - Frequency Shift: $e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
    - Time Reversal: $x[-n] \leftrightarrow X(e^{-j\omega})$
    - Conjugate Symmetry - $x^*[n] \leftrightarrow X^*(e^{-j\omega})$
      - even, real ⇒ all real
      - odd, real ⇒ all imaginary
    - Time Expansion $x_{(M)}[n] = \begin{cases} x[n/M] & \text{if } n \text{ mod } M \\ 0 & \text{otherwise} \end{cases}$
      $x_{(M)}[n] \leftrightarrow X(e^{j\omega M})$
    - Differentiation $nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$ $\quad$ ← $\omega=0$ just a flip
      $x[n]-x[n-1] \leftrightarrow (1-e^{-j\omega})X(e^{j\omega})$ $\quad$ ← draw pics
    - Parseval's Relation $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2d\omega$ $\quad$ periodic ← convolution
    - Multiplication Property $x_1[n]\cdot x_2[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$
    - Convolution Property $(x_1*x_2)[n] \leftrightarrow X_1(e^{j\omega})\cdot X_2(e^{j\omega})$
    - $H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_ke^{-j\omega k}}{\sum_{k=0}^{N} a_ke^{-j\omega k}}$ $\quad \sum_{k=0}^{N} a_ky[n-k] = \sum_{k=0}^{M} b_kx[n-k]$
  - $\text{sinc}(n) = \begin{cases} \frac{\sin(\pi n)}{\pi n} & n\neq 0 \\ 1 & n=0 \end{cases}$ ← 0 at integers



- convolution of rectangles

- Tip, get DTFS / CTFS coeff from synthesis equation, just expand
- Convergence Tests
  - Integral Test $\int_0^\infty f(x)dx$ converges
  - Comparison Test - find something bigger
  - Ratio Test $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L < 1$ converges
  - Root Test $\lim_{k\to\infty} (a_k)^{1/k} = L < 1$ converges

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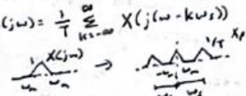
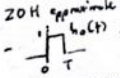
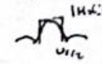
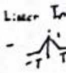
EE120
MT2

- Discrete Fourier Transform (DFT)
  - $x[n]$ finite length N sequence
  - $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$, $k = 0,1,\dots,N-1$
  - $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$, $n = 0,1,\dots N-1$
  - Properties
    - conjugate symmetry $X^*[N-k] = X[k]$, $k = 1,2,\dots N-1$
    - connection to DTFT $X[k] = X(e^{j\omega})|_{\omega=\frac{2\pi}{N}k}$
    - convolution $y[n] = h[n] * x[n] \leftrightarrow H[k]X[k]$, just pad 0s
- 2D CTFT
  - $X(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2) e^{-j\omega_1 t_1} e^{-j\omega_2 t_2} dt_1 dt_2$
  - $x(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega_1,j\omega_2) e^{j\omega_1 t_1} e^{j\omega_2 t_2} d\omega_1 d\omega_2$
  - absolute integrability condition for convergence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t_1,t_2)| dt_1 dt_2 < \infty$
- 2D DTFT
  - $X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x[n_1,n_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$ ← 2π repeating
  - $x[n_1,n_2] = \frac{1}{(2\pi)^2}\int_{2\pi}\int_{2\pi} X(e^{j\omega_1},e^{j\omega_2}) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$
  - abs sum cond for convergence $\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} |x[n_1,n_2]| < \infty$
  - $\delta[n_1,n_2] = \delta[n_1]\delta[n_2]$
  - separability property $X(e^{j\omega_1}, e^{j\omega_2}) = X_1(e^{j\omega_1}) \cdot X_2(e^{j\omega_2})$
  - convolution $h[n_1,n_2] * x[n_1,n_2] = H(e^{j\omega_1},e^{j\omega_2}) X(e^{j\omega_1},e^{j\omega_2})$
- 2D DFT
  - $X[k_1,k_2] = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x[n_1,n_2] e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2}$
  - $x[n_1,n_2] = \frac{1}{N_1 N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} X[k_1,k_2] e^{j\frac{2\pi}{N_1}k_1 n_1} e^{j\frac{2\pi}{N_2}k_2 n_2}$
- Sampling
  - Discrete Time sequence of CT signal, $x_d[n] = x(nT)$ T: sampling period
  - Shannon-Nyquist Sampling Theorem (bandlimited signal, use LPF or nothing)
    - $\omega_s > 2\omega_M$ ⇒ can reconstruct signal always, strictly greater ⇒ $x_r(t) = x(t)$
    - $x_p(t) = x(t) \cdot p(t)$, $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ ...↑↑↑↑... $-T\ 0\ T\ 2T$
    - For $\delta(t)$, $a_k = \frac{1}{T}$, $P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$
    - $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$ ✓
    - 

      shifted copies
    - if $\omega_s < 2\omega_m$, then shifts overlap, get "aliasing"
      - high freq look like lower freq
    - Reconstruction filter $H_r(j\omega)$: T from $-\frac{\omega_s}{2}$ to $\frac{\omega_s}{2}$ (and amplify) $\omega_s = \frac{2\pi}{T}$
  - Ideal → $h_r(t) = T\frac{\omega_s}{2\pi}\operatorname{sinc}(\frac{\omega_s}{2\pi}t) = \operatorname{sinc}(\frac{t}{T})$ 0s at T, 2T, ...
    - $x_r(t) = h_r(t) * x_p(t) = \sum_n x(nT) h_r(t - nT)$
      $= \sum_{n=-\infty}^{\infty} x(nT)\operatorname{sinc}(\frac{t-nT}{T})$
      - to reconstruct, sum of bunch of shifted, scaled sincs
- ZOH approximate reconstruction
  - 

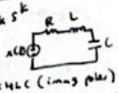
    rec. $H_0(j\omega) = e^{-j\omega T/2}\, T\operatorname{sinc}(\frac{T}{2\pi}\omega)$
  - 

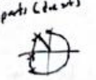
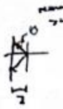
    not bad except for higher freq
- Linear Interpolation
  - 

    $H_1(j\omega) = T\operatorname{sinc}^2(\frac{T}{2\pi}\omega)$
- Wagon Wheel Effect - appear rotate slowly backward if equal framerate
  - sometimes like $\omega_{appear} + \omega_{sample}k = \omega_{actual}$
- Critical Freq - $\omega_s/2$, T/2 at $\omega_s/2$ b/c peaks add
  - ex. $x(t) = \cos(\omega_0 t + \phi)$ $\omega_s = 2\omega_0$ ⇒ $x_r(t) = \cos(\phi)\cos(\omega_0 t)$
- DSP $x(t) \to \boxed{C/D} \xrightarrow{x_d[n]} \boxed{H_d} \xrightarrow{y_d[n]} \boxed{D/C} \to y(t)$
  - $\Omega = \omega T$ $X_d(e^{j\Omega})$ $Y_d(e^{j\Omega})$ Ω in radians $X_d(e^{j\Omega})|_{\Omega=\omega T} = X_p(j\omega)$
  - $Y(\omega) = H_d(e^{j\omega T}) X(\omega)$ if $|\omega| < \omega_s/2$ $Y(\omega) = T H_d(e^{j\omega T}) X_p(\omega)$
  - Remember $\delta[n] \leftrightarrow 1$ $\delta[n - n_0] \leftrightarrow e^{-j\Omega n_0}$
  - ex. implement delay $h_d[n] = \operatorname{sinc}(n - \frac{\Delta}{T})$ $y(t) = x(t - \Delta) \to H(\omega) = e^{-j\omega\Delta} = e^{-j\frac{\Omega}{T}\Delta}$
  - Just know $H_d(e^{j\Omega})$ is freq response of DT system
  - Also $\operatorname{sinc}(n) = \frac{\sin(\pi n)}{\pi n}$ and $\operatorname{sinc}(an) \leftrightarrow$ $\frac{1}{a}$ from $-a\pi$ to $a\pi$
  - Downsampling - every Nth signal
    $x_{\downarrow N}[n] = x[Nn]$
  - Upsample: spread out, add 0s, apply sinc
    $y_{\uparrow N}[n] = y[n/N]$ $n = 0, \pm N, \dots$
    0 o/w

- Laplace Transform $s = \sigma + j\omega$
  - $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$, $s \in \mathbb{C}$
    - if $s = j\omega$, then get FT; $\mathcal{L}\{x(t)\} = \mathcal{F}\{x(t)e^{-\sigma t}\}$
  - Region of Convergence (ROC) - set $s \in \mathbb{C}$ where σ is such that Fourier Int. for $x(t)e^{-\sigma t}$ converges, if ROC includes imag axis ($\sigma = 0$) then FT exists for $x(t)$
  - Poles and Zeros
    - $X(s) = \frac{N(s)}{D(s)}$ zeros - $N(s) = 0$ o; poles - $D(s) = 0$ x
  - Inverse Laplace Transform by PFE
    - match ROC, split up, pattern match - ex. $\frac{1}{(s+1)^2} \to \frac{A}{s+1} + \frac{B}{(s+1)^2}$
  - Properties of LT $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$, ROC = R
    - Linearity - $ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s)$
      - ROC = $R_1 \cap R_2$
    - Time shift - $x(t - t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0}X(s)$ ROC same
    - s-shift - $e^{s_0 t}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s - s_0)$
      - ROC = R + Re{s_0}
    - Time scale - $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|}X(\frac{s}{a})$ ROC = aR
    - Conjugation $x^*(t) \overset{\mathcal{L}}{\leftrightarrow} X^*(s^*)$ ROC same
    - Convolution $(x_1 * x_2)(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)X_2(s)$ ROC = $R_1 \cap R_2$
    - Diff in time $\frac{d}{dt}x(t) \overset{\mathcal{L}}{\leftrightarrow} sX(s)$ ROC has R, bigger/larger
    - Diff in s $-tx(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{d}{ds}X(s)$ ROC same for exponential
    - Integration in time $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}X(s)$ ROC has $R \cap \{s : Re\{s\} > 0\}$
  - Initial Value Theorem - if $x(t) = 0$ for $t < 0$ no impulse or singularity at $t = 0$, then $x(0^+) = \lim_{s\to\infty} sX(s)$
  - Final Value Theorem - if $x(t) = 0$ for $t < 0$, $x(t)$ finite limit t→∞,
    $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$
  - Common Transforms

		ROC
$\delta(t)$	1	all s
$u(t)$	$\frac{1}{s}$	Re{s} > 0
$-u(-t)$	$\frac{1}{s}$	Re{s} < 0
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	Re{s} > 0
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	Re{s} < 0
$e^{-at}u(t)$	$\frac{1}{s+a}$	Re{s} > -a
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	Re{s} < -a
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	Re{s} > -a
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	Re{s} < -a
$\delta(t-T)$	e^{-sT}	all s
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	Re{s} > 0
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	Re{s} > 0
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	Re{s} > -a
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	Re{s} > -a

- Transfer Function of LTI
  - $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$
  - $H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$
  - $I = C\frac{dV}{dt}$, $V = L\frac{dI}{dt}$ 

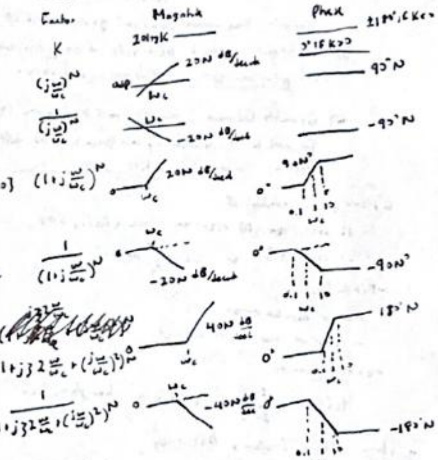
    - oscillation if $R^2C^2 < 4LC$ (imag poles)
- $Y(s) = H(s)X(s)$ (causal LTI)(H(s) rational) left of imag
  - stable iff all poles of H(s) strictly negative real parts (due ↙)
- Butterworth Filter - $|H(j\omega)|^2 = \frac{1}{(1+(\omega/\omega_c)^{2N})}$ 
- Second order system $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta s + \omega_n^2}$
  - ζ is damping ratio, resonance if $\zeta < \frac{1}{\sqrt{2}}$
  - always has complex conjugate poles if $\zeta < 1$
  - $s = \omega_n(-\cos\theta \pm j\sin\theta)$ $\zeta = \cos\theta$ 

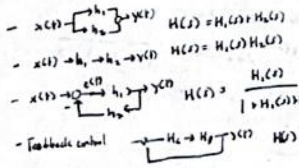
- 2D sampling
  - $X_p(\omega_1,\omega_2) = \frac{1}{T_1T_2}\sum_{k_1}\sum_{k_2} X(\omega_1 - k_1\omega_{s_1}, \omega_2 - k_2\omega_{s_2})$
  - $|\omega_{s_1}| > 2\omega_{m_1}$ ... same as 1D.

- Bode Plots
  - dB scale - $20\log_{10}|H(j\omega)|$
  - $H(s) = K\frac{(\tau_1 s+1)\dots(\tau_M s+1)}{(\tau_{M+1}s+1)\dots(\tau_{M+N}s+1)}$

$20\log_{10}|H(j\omega)| = 20\log_{10}|K| + \sum_{i=1}^{M} 20\log_{10}|j\omega\tau_i + 1|$
$- \sum_{i=M+1}^{M+N} 20\log_{10}|j\omega\tau_i + 1|$

  - construct subplots and sum up
  - $\angle H(j\omega) = \sum_{i=1}^{M}\angle(j\omega\tau_i + 1) - \sum_{i=M+1}^{M+N}\angle(j\omega\tau_i + 1)$
  - Common Plots





- parallel: $H(s) = H_1(s) + H_2(s)$
- $x(t) \to h_1 \to h_2 \to y(t)$ $H(s) = H_1(s)H_2(s)$
- feedback: $H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$
- Feedback control $H(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$
- $\frac{d}{dt}\vec{z}(t) = A\vec{z}(t) + Bx(t)$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
  $y(t) = C\vec{z}(t) + Dx(t)$
  $H(s) = C(sI - A)^{-1}B + D$ ← poles are λ of A
- Step Response of 2nd order system $u(t) \leftrightarrow \frac{1}{s}$ $H(s) = \frac{\omega_n^2}{s^2 + 2\zeta s\omega_n + \omega_n^2}$
  - Rise Time (t_r) - 10% to 90% 
  - Peak overshoot (M_p) - (peak - steady)/steady
  - Peaking time (t_p) - time to M_p
  - Settling time (t_s) - time after within 1% of steady state
- poles $s = \underbrace{-\omega_n\cos\theta}_{\sigma} \pm j\underbrace{\omega_n\sin\theta}_{\omega_d}$ $\cos\theta = \zeta$
- $y(t) = (1 - (\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t)e^{-\sigma t})u(t)$
  - $t_p = \pi/\omega_d$ $t_s \approx \frac{4.6}{\sigma} = \frac{4.6}{\omega_n\cos\theta}$
  - $M_p = e^{-\pi\frac{\sigma}{\omega_d}} = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}$ $t_r \approx \frac{1.8}{\omega_n}$
- Unilateral Laplace Transform
  - $\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$
  - convolution - useless, follows from bilateral
  - diff in time $\frac{d}{dt}x(t) \leftrightarrow s\mathcal{X}(s) - x(0^-)$
    $\frac{d^2}{dt^2}x(t) \leftrightarrow s^2\mathcal{X}(s) - sx(0^-) - \frac{d}{dt}x(0^-)$
  - good for solving diff eq w/ init cond
- Feedback control $H(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$ we control $H_c(s)$
  - constant gain $H_c(s) = K$ poles at $1 + KH_p(s) = 0$

- More Feedback Control Stuff
  - Root Locus Analysis
    - How do roots change as we move constants? (poles) $1+KH(s)=0$?
    - $H(s)=\frac{\prod_{k=1}^{m}(s-\beta_k)\text{ roots}}{\prod_{k=1}^{n}(s-\alpha_k)\text{ poles}}$ $m\le n$

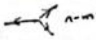
      if s_0 on root locus then $H(s_0)=-\frac{1}{K}$ for $K>0$

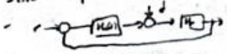
      $\angle H(s_0)=\pi$

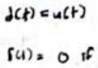
    1) As $K\to 0$ roots converge to poles of $H(s)$ $H(s)\to\infty$
       - n poles → n branches → each start at pole of $H(s)$
    2) As $K\to\infty$, m branches approach zeros of $H(s)$
       - if $m<n$, then n-m branches approach ∞ following asymptote at

$$\frac{\sum_{k=1}^{n}\alpha_k-\sum_{k=1}^{m}\beta_k}{n-m}$$

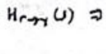
w/ angles $\frac{180^\circ+(i-1)360^\circ}{n-m}$ $i=1,2,\ldots n-m$

       - basically they approach zeros and go out and split off
    3) Parts of real line that lie to left of an <u>odd</u> number of real <u>poles and zeros</u> of $H(s)$ are on root locus
    4) Branches between 2 real poles must break away into complex plane for some $K>0$. Breakaway and break-in points determined by solving for roots of $\frac{dH(s)}{ds}=0$ that on real line
  - High gain instability if
    - 1) $H(s)$ zeros in right half plane ($Re\{s\}>0$)
    - 2) $n-m\ge 3$ ex  $n-m=3$
  - Design Feedback
    - draw desired region
    - draw root locus see if intersect
  - Lead Controller

    $H_c(s)=K\frac{s-\beta}{s-\alpha}$ $\alpha<\beta<0$ has phase lead
- Steady State Tracking Accuracy
  - e_{ss} = error steady state, $t\to\infty$
  - $e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\frac{1}{1+H_c(0)H_p(0)}$
  - For $e_{ss}=0$, need $\lim_{s\to 0}H_c(s)H_p(s)=\infty$
    - one pole at $s=0$
  - Integral Control
    - 0 steady state error by def $\frac{1}{s}$
    - slower response, harder to tune
    - + lead control $H_c(s)=\frac{K}{s}\frac{s-\beta}{s-\alpha}$ similar to PID
  - Disturbance Rejection
    - like if I push it or wind
    - 

    - $Y(s)=\underbrace{\frac{H_c(s)H_p(s)}{1+H_c(s)H_p(s)}R(s)}_{\text{normal}}+\underbrace{\frac{H_p(s)}{1+H_c(s)H_p(s)}D(s)}_{\Delta(s)}$
    - assume $d(t)=u(t)$ 

      - $\lim_{t\to\infty}\delta(t)=0$ if $H_c(s)$ has pole at $s=0$
      - integral does that ↗
    - $H_{d\to y}(s)=\frac{H_p(s)}{1+H_c(s)H_p(s)}$, $H_{r\to y}(s)=\frac{H_c(s)H_p(s)}{1+H_c(s)H_p(s)}$
      - want $H_{r\to y}(s)\to 1$, so $|H_c(s)H_p(s)|>>1$
      - want $H_{d\to y}(s)\to 0$, so $|H_c(s)|>>1$ (pole at $s=0$)
  - Noise Insensitivity



    - $H_{n\to y}(s)=H_{r\to y}(s)$ ⇒ can't easily take it out, but maybe use a filter or something

- Z transform
  - [illegible]
  - $X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$
  - $X(z)|_{z=e^{j\omega}}=X(e^{j\omega})$ DTFT
    - converges if ROC includes unit circle

       $e^{j\omega}$ b/c $|e^{j\omega}|=1$
  - Properties of ROC
    - ring or disk
    - no poles
    - $x[n]$ right sided, extends from outermost pole to ∞
  - Inverse Z Transform by PFE

    $X(z)=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}$ $a_0\ne 0$
    - unrepeated poles $d_1,d_2,\ldots d_N$
    - $M<N$ $X(z)=\sum_{k=1}^{N}\frac{A_k}{1-d_kz^{-1}}$ ↴
    - $M\ge N$ $X(z)=\sum_{r=0}^{M-N}B_rz^{-r}+\sum_{k=1}^{N}\frac{A_k}{1-d_kz^{-1}}$

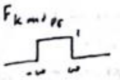
      $x[n]=\sum_{r=0}^{M-N}B_r\delta[n-r]+\sum_{k=1}^{N}A_kd_k^{n}u[n]$
  - differentiation in z

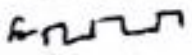
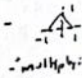
    $x[n]\overset{z}{\leftrightarrow}X(z)$

    $nx[n]\leftrightarrow -z\frac{d}{dz}X(z)$
  - Properties
    - Linearity: $ax_1[n]+bx_2[n]\overset{z}{\leftrightarrow}aX_1(z)+bX_2(z)$ ROC $=R_1\cap R_2$
    - Time shift $x[n-n_0]\overset{z}{\leftrightarrow}z^{-n_0}X(z)$ ROC same (maybe 0, ∞)
    - Scale $z_0^{n}x[n]\overset{z}{\leftrightarrow}X(\frac{z}{z_0})$ ROC $=|z_0|\cdot R$
    - Time Reversal $x[-n]\overset{z}{\leftrightarrow}X(\frac{1}{z})$ ROC $=1/R$
    - convolution $x_1[n]*x_2[n]\overset{z}{\leftrightarrow}X_1(z)X_2(z)$ ROC $=R_1\cap R_2$ contains
  - initial IVT $x[0]=\lim_{z\to\infty}X(z)$ if $x[n]=0$ for $n<0$
  - Common Transform

		ROC
$\delta[n]$	1	all z
$\delta[n-m]$	z^{-m}	all z except $z=0$ if $m>0$ all z except $z=\infty$ if $m<0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>a$
$-a^nu[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert<a$
$+na^nu[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert>a$
$-na^nu[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert<a$
$\cos(\omega_0n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$\sin(\omega_0n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert>1$
$r^n\cos(\omega_0n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$
$r^n\sin(\omega_0n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert>r$

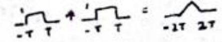
- Extra
  - LTI out is convolution w/ impulse response
  - Sum of periodic signal over period = 0
  - LTI can't introduce new frequencies, only modify them
    - period at most that of input
  - FS coeff of constant are constant
  - Find $H(j\omega)$ b/c take DTFT of both sides
  - $\omega_0=\frac{2\pi}{T}$ = fundamental frequency
  - upscaling by N ⇒ FS coeff scaled by $1/N$

    $G_k=\frac{1}{N}F_{k\bmod N}$
  - $x(t)=\frac{W}{\pi}\text{sinc}(\frac{W}{\pi}t)\overset{FT}{\leftrightarrow}$ 

  - $\sum_{x=0}^{\infty}a^x=\frac{1}{1-a}$ $\sum_{k=0}^{n}a^k=\frac{1-a^n}{1-a}$
  - $h[n]=a^nu[n]\leftrightarrow H(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$
  - stability $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$
- real
  - even signal → all real FS
  - odd signal → all imag FS
  - $\sum_{k=\langle N\rangle}a_k=1$ $\sum_{k=\langle N\rangle}(-1)^ka_k=0$ for 
  - remember synthesis equation to find sum
  -  ↔ $\text{sinc}^2(\frac{\omega}{2\pi})$

  - "multiplying" signals = mult Fourier Transform
  - $\int_{-\infty}^{\infty}\text{sinc}^2(t)dt=1$ ⟺ Parseval's relation
  - PID $x(t)=K_pe(t)+K_i\int_{-\infty}^{t}e(t)+K_d\frac{d}{dt}e(t)$

    $H_c(s)=K_p+K_i\cdot\frac{1}{s}+K_d\cdot s$
  - unit response = convolution w/ $u(t)$ and impulse response



$g(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}}\overset{FT}{\leftrightarrow}G(\omega)=e^{-\frac{\omega^2\sigma^2}{2}}$

  - $\delta(n-N)\overset{FT}{\leftrightarrow}e^{-j\omega N}$
  - To get rid of something just $H(\omega)G(\omega)=1$
  - split abs value
  - Parseval's Thm $\int_{-\infty}^{\infty}|v(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2d\omega$
  - periodic ⇒ $2\pi k$ + phaseshift time
  - $\frac{d}{dt}x(t)\leftrightarrow j\omega X(\omega)$
  - remember reconstruction filter gain T
  - For $\frac{\omega_n^2}{s^2+2\zeta\omega_ns+\omega_n^2}$, $s=-\omega_n\zeta\pm\omega_n\sqrt{\zeta^2-1}$ poles

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EE120
Final

- DT LTI
  - $Y(z) = H(z)X(z)$
  - $x[n] \to \boxed{h[n]} \to y[n] = (h * x)[n]$
  - $H(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$
  - causal DT LTI w/ H(z) stable iff all poles of H(z) in unit circle
  - Difference Equation to Transfer Function
    - $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

$$\Updownarrow$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

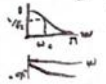
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

  - Geometric Eval of Frequency Response $H(e^{j\omega})$

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \begin{matrix} \text{zeros} \\ \text{poles} \end{matrix}$$

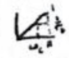
$$|1 - d_k e^{-j\omega}| = |e^{j\omega} - d_k| = |V_1| \text{ vector approach}$$

  - Low Pass
    - $H(z) = \frac{1-\alpha}{2} \frac{1+z^{-1}}{1+\alpha z^{-1}}$ $|\alpha| < 1$ for stability



~~where~~ $\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$

  - High Pass
    - $H(z) = \frac{1+\alpha}{2} \frac{1-z^{-1}}{1-\alpha z^{-1}}$

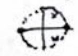


$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$

  - Band Stop (Notch)
    - $H(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha) z^{-1} + \alpha z^{-2}}$ $|\beta| < 1$ $|\alpha| < 1$

$\beta = \cos \omega_0$
poles approach zeros as $\alpha \to 1$
↗
sharper cutoff

  - Band Pass
    - $H(z) = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1+\alpha) z^{-1} + \alpha z^{-2}}$ $|\alpha| < 1$ $|\beta| < 1$



$\beta = \cos \omega_0$

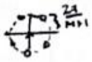
sharper as $\alpha \to 1$

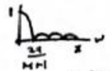
  - (M+1) point moving average filter

$$y[n] = \frac{1}{M+1} (x[n] + x[n-1] + \dots + x[n-M])$$

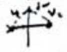
$$H(z) = \frac{1}{M+1} \frac{z^M + z^{M-1} + \dots + 1}{z^M}$$

    - all poles at z=0 (true for any FIR system)

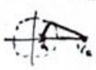




  - All Pass Filters (Phase Compensator)
    - CT $H(s) = \frac{a-s}{s+a}$



    - DT $H(z) = \frac{z^{-1} - a}{1 - a z^{-1}} = -a \frac{z - 1/a}{z - a}$, a real



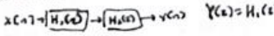
basically every pole c_k has zero $\frac{1}{c_k^*}$

- Unilateral Z-transform
  - $\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$
  - convolution: $(x_1 * x_2)[n] \overset{UZ}{\leftrightarrow} \mathcal{X}_1(z)\mathcal{X}_2(z)$ if $x_1[n] = x_2[n] = 0 \ \forall n < 0$
  - time delay $x[n-1] \overset{UZ}{\leftrightarrow} z^{-1}\mathcal{X}(z) + x[-1]$

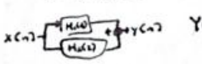
$$x[n-2] \overset{UZ}{\leftrightarrow} z^{-2}\mathcal{X}(z) + z^{-1}x[-1] + x[-2$$

$$\vdots$$

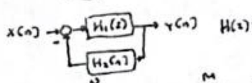
  - helps solve difference equations
    - take UZ of both sides
    - solve for Y(z), use partial fraction decomp to get y[n]
  - Interconnection of DT LTI Systems



$Y(z) = H_1(z)H_2(z)X(z)$

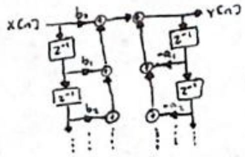


$Y(z) = (H_1(z) + H_2(z))X(z)$



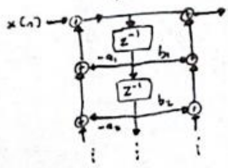
$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$

  - $y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$



N+M delay elements
(memory registers)

⇓ can merge



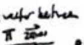
  - Transfer Functions from State Space Models

$$\vec{\omega}[n+1] = A\vec{\omega}[n] + Bx[n]$$

$$y[n] = C\vec{\omega}[n] + Dx[n]$$

$$H(z) = C(zI - A)^{-1}B + D$$

poles of H(z) are eigenvalues of A
- label each delay w/ $\omega_i[n]$
- write equation for each

- Extra
  - To draw magnitude (or phase) diagram
  - 0 to π is just  on unit circle
  - take vector between 0s and poles to point
    - $\frac{\prod \overrightarrow{zeros}}{\prod \overrightarrow{poles}}$ or $\sum \angle \overrightarrow{zeros} - \sum \angle \overrightarrow{poles}$
  - causal & rational ⇒ ROC outside outermost pole
  - Accumulation Property

$$\tilde{x}[n] = \sum_{k=-\infty}^{n} x[k] \leftrightarrow \tilde{X}(z) = \frac{1}{1-z^{-1}} X(z) \quad \text{ROC at least } R \cap |z|>1$$

  - causal ⇒ ROC outside outermost pole
  - stable ⇔ ROC contains unit circle
  - PFE is ur friend
  - ZIR zero input response
  - unilateral z transform for initial condition stuff
  - Kronecker delta δ[n]
  - Fractional DT delay done w/ sinc
  - $e^{-\alpha t^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$
  - Gaussian check inter multiplication and convolution
  - $e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$
  - Gaussian FT def use derivative prop
  - Fourier Transform Sampling

$$x_p(t) = x(t) \cdot p(t)$$

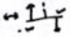
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \text{ impulse train T apart}$$

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\overset{\omega - k\omega_s}{\cancel{\omega - k\omega_s}})$$

↗
FS coeff of impulse train are $a_k = \frac{1}{T}$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$\omega_s = \frac{2\pi}{T}$

  - bandwidth = (big freq − small freq)
  - $\Omega = \omega T$
  - $\cos(\omega_0 t) \leftrightarrow$  $\sin(\omega_0 t) \leftrightarrow$ 
  - δ is unbounded